GCSE Higher Tier Mathematics

Circle Theorem Proofs

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Instructions

- Answer all questions in the spaces provided.
- You must show every step of your logic. A ”proof” is a logical argument, not just a calculation.
- Use algebraic letters (like x, y, or θ) to represent unknown angles.
- State the geometrical reason for every step you take (e.g., ”Base angles of isosceles △”).

Key Concepts: The Toolkit for Proofs

Deeper Insight: What is a Proof?

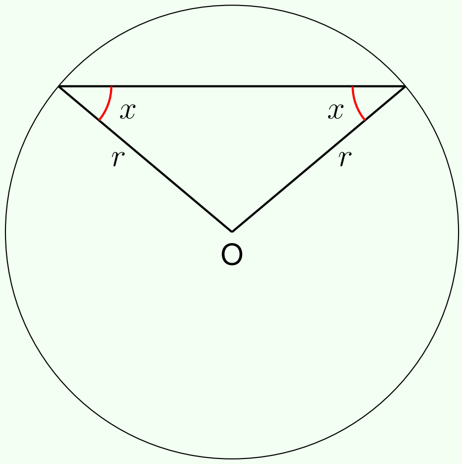
In mathematics, checking a specific case (e.g., ”it works for $30°$”) is **not** a proof. To prove a theorem, you must show it is true for **any** angle. We do this by using algebra (letting an angle be x) and logical steps that are always true (like ”angles on a straight line add to $180°$”).

Method: Your Two Hidden Weapons

Almost every Circle Theorem proof relies on two simple facts that are often hidden in the diagram.

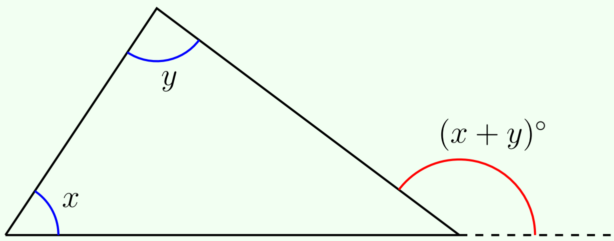
1. Radii create Isosceles Triangles

Any two lines from the centre to the circumference are the same length (radius). If you join them with a chord, you create an isosceles triangle with equal base angles.



2. Exterior Angle of a Triangle

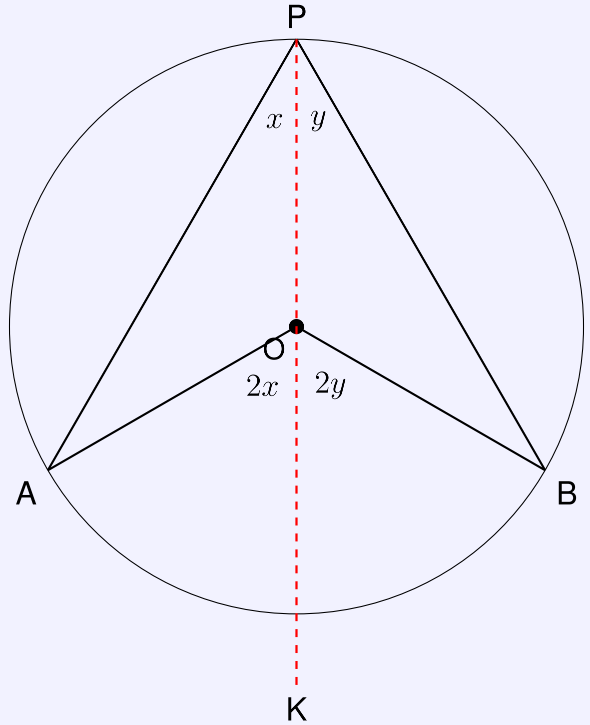
The exterior angle of a triangle is equal to the sum of the two opposite interior angles. This is often faster than finding the third angle.



Theorem 1: Angle at Centre is Twice Angle at Circumference

Guided Proof: The "Arrowhead" Case

Goal: Prove that if $\angle APB = \theta$, then $\angle AOB = 2\theta$.



Step 1: Create Triangles

Draw a dashed line from P through O to a point K. This splits the shape into two triangles, $\triangle APO$ and $\triangle BPO$.

Step 2: Use Isosceles Properties

Consider $\triangle APO$. Since OA and OP are both radii, the triangle is isosceles. Therefore, $\angle OAP = \angle OPA = x$. Similarly, for $\triangle BPO$, $\angle OBP = \angle OPB = y$.

Step 3: Exterior Angles

Look at $\angle AOK$ (the exterior angle of $\triangle APO$).

$$\angle AOK = \angle OAP + \angle OPA = x + x = 2x$$

Similarly, $\angle BOK = \angle OBP + \angle OPB = y + y = 2y$.

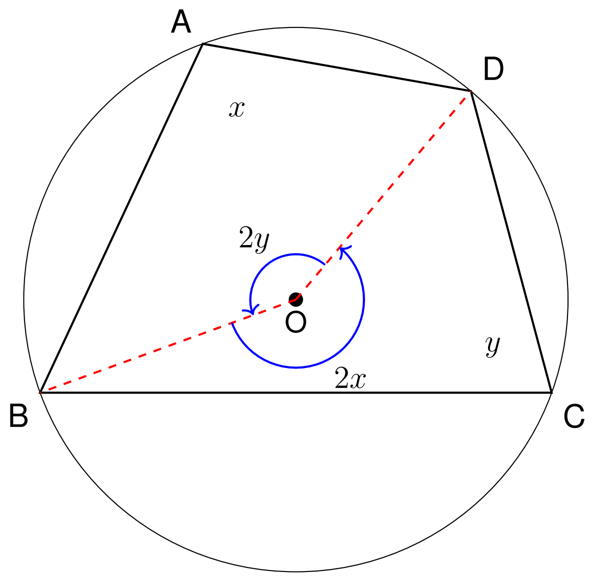
Step 4: Conclusion

Total Angle at Centre $(\angle AOB) = 2x + 2y = 2(x + y)$. Total Angle at Circumference $(\angle APB) = x + y$. Therefore:

$$\angle AOB = 2 \times \angle APB$$

Q.E.D.

1. **Guided Proof: Cyclic Quadrilaterals**
   Prove that the opposite angles of a cyclic quadrilateral sum to $180°$.



Step 1: Draw radii from the centre O to points B and D.
Step 2: Use the "Angle at Centre" theorem.

(a) Consider the angle x at A. The angle at the centre subtended by arc BCD is reflex angle BOD.
Write an expression for reflex $\angle BOD$ in terms of x:

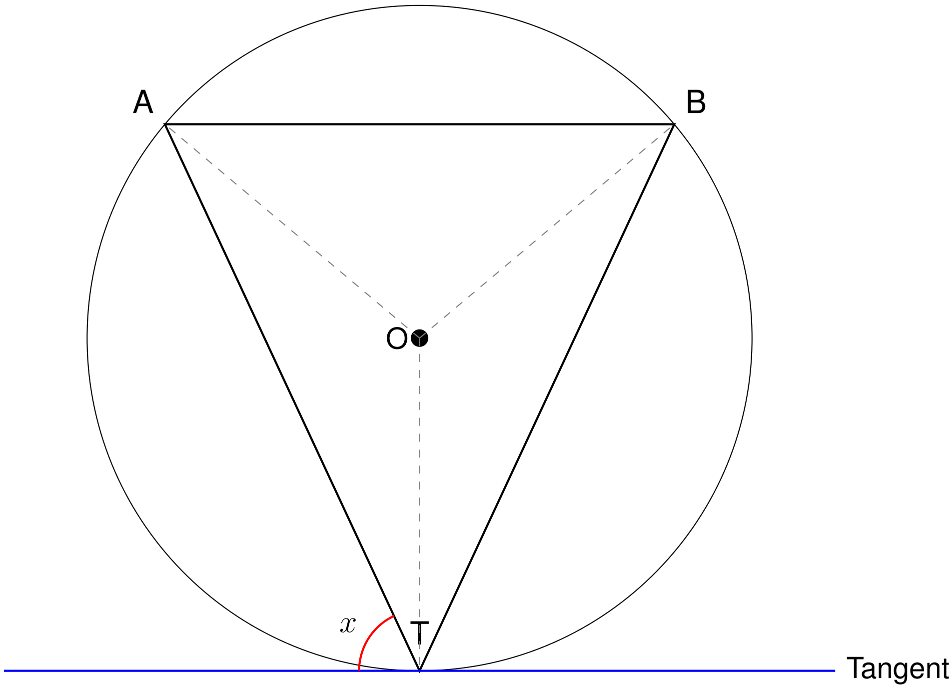
(b) Consider the angle y at C. The angle at the centre subtended by arc BAD is obtuse angle BOD.
Write an expression for obtuse $\angle BOD$ in terms of y:

Step 3: Angles around a point.

(c) The angles around the centre O must add up to $360°$. Write an equation connecting your answers to (a) and (b).

(d) Simplify your equation to prove that $x + y = 180°$. [2]

2. **Standard Proof: The Alternate Segment Theorem**
Prove that the angle between a tangent and a chord is equal to the angle in the alternate segment.



Given: A tangent at T and a triangle ABT. Let the angle between the tangent and chord AT be x.

Goal: Prove that $\angle ABT = x$.

Step 1: **Tangent Property:** Explain why $\angle OTA = 90 - x$.

Step 2: **Isosceles Triangle:** Consider $\triangle OAT$. Explain why $\angle OAT = 90 - x$.

Step 3: Angle Sum: Calculate $\angle AOT$ in terms of x (using the angles in $\triangle OAT$).

Step 4: Circle Theorem: Use the ”Angle at Centre” theorem to find $\angle ABT$.

Step 5: Conclusion: State your final proof clearly.

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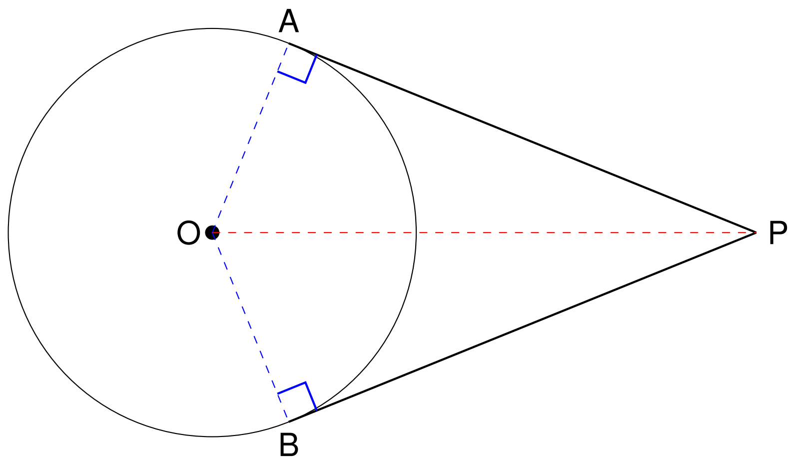
3. **Congruence Proof: Tangent Lengths**
Two tangents are drawn from an external point P to touch the circle at points A and B. O is the centre of the circle. Prove that the lengths PA and PB are equal.



Use the properties of congruent triangles to complete the proof.

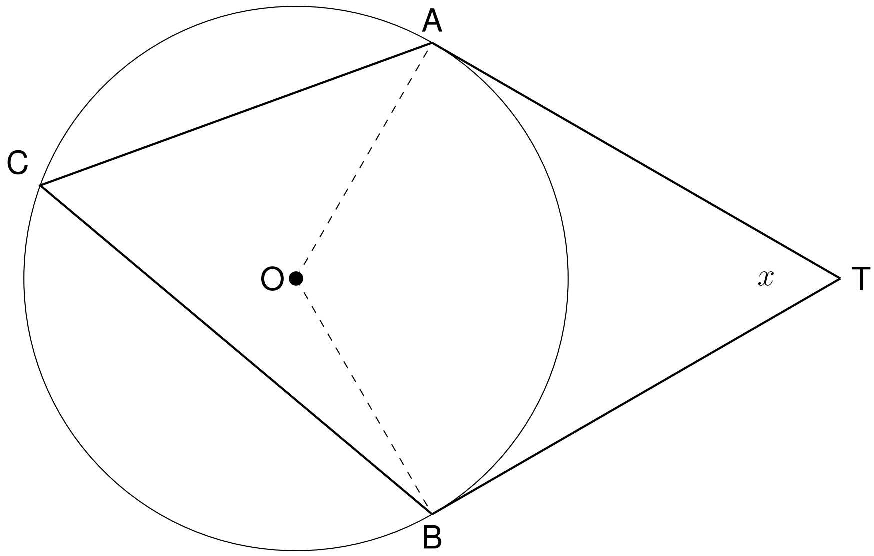
(a) Compare $\triangle OAP$ and $\triangle OBP$. State three properties that are equal for both triangles and give the reason for each.

(b) State the condition of congruence (e.g., SSS, SAS, ASA, RHS) that applies here.

(c) Hence, prove that $PA = PB$.

4. **Exam Style Application**
   A, B, and C are points on a circle with centre O. Tangents are drawn at A and B and meet at an external point T.

   Prove that $\angle ACB = 90^\circ - \frac{1}{2}\angle ATB$.



   Hint: Let $\angle ATB = x$. Start by finding angle AOB.

This is the end of the worksheet

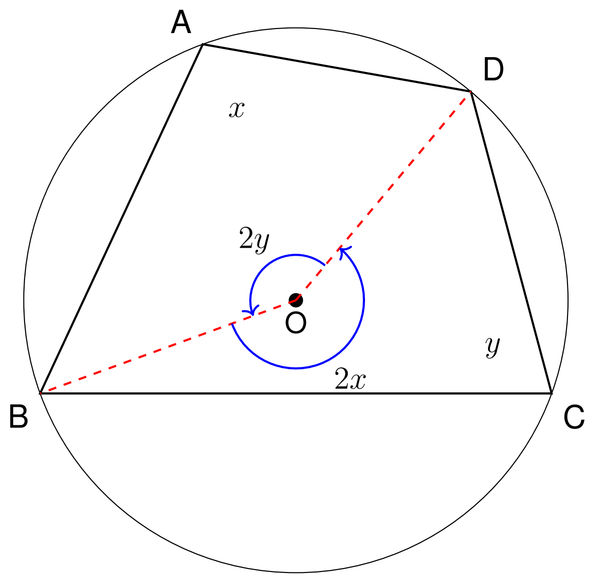
Worked Solutions & Proofs

Deeper Insight: How to Write a Proof

In the exam, your "reasons" are as important as your algebra. You cannot just write "$x + y = 180$". You must write "$x + y = 180$ because angles at a point sum to $360°$".

The examiners are looking for the **keywords**: "Radii", "Isosceles", "Tangent", "Angle at Centre".

1) Proof: Opposite Angles of a Cyclic Quadrilateral



(a) **Reflex Angle BOD:** Angle x at the circumference subtends the major arc.

$$\text{Reflex } \angle BOD = 2 \times \angle BAD = \mathbf{2x}$$

(b) **Obtuse Angle BOD:** Angle y at the circumference subtends the minor arc.

$$\text{Obtuse } \angle BOD = 2 \times \angle BCD = \mathbf{2y}$$

(c) **Equation:** Angles around a point sum to $360°$.

$$\text{Reflex } \angle BOD + \text{Obtuse } \angle BOD = 360°$$

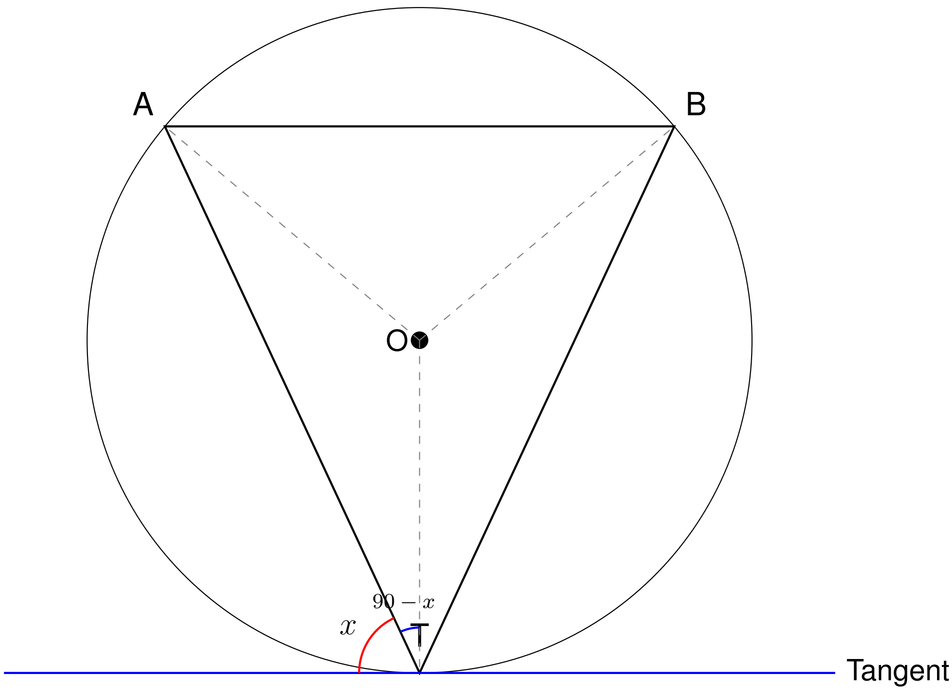
$$\mathbf{2x + 2y = 360°}$$

(d) **Conclusion:** Divide the entire equation by 2:

$$x + y = 180°$$

Therefore, the opposite angles sum to $180°$. **Q.E.D.**

2) Proof: The Alternate Segment Theorem



Step 1: Tangent Property: The radius OT is perpendicular to the tangent at point T.

$$\angle OTA = 90^\circ - x$$

Reason: Tangent meets radius at 90°*.*

Step 2: Isosceles Triangle: In $\triangle OAT$, $OA = OT$ (both are radii). Therefore, the triangle is isosceles.

$$\angle OAT = \angle OTA = 90^\circ - x$$

Reason: Base angles of isosceles triangle.

Step 3: Angle Sum: Angles in a triangle sum to 180°.

$$\angle AOT = 180^\circ - (90 - x) - (90 - x)$$
$$\angle AOT = 180 - 90 + x - 90 + x$$
$$\angle AOT = \mathbf{2x}$$

Step 4: Circle Theorem: The angle at the circumference ($\angle ABT$) is half the angle at the centre ($\angle AOT$).

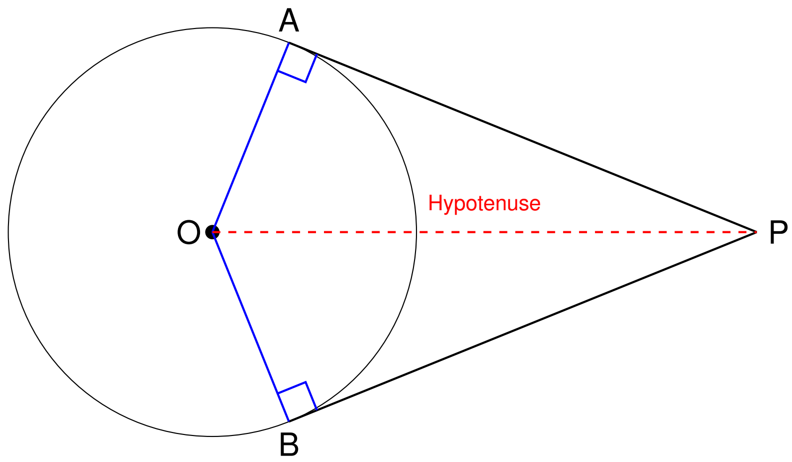
$$\angle ABT = \frac{1}{2} \times \angle AOT = \frac{1}{2}(2x) = x$$

Step 5: Conclusion: We defined the angle between chord and tangent as x, and proved the angle in the alternate segment is also x.

$$\angle ABT = x$$

Q.E.D.

3) Proof: Tangent Lengths from External Point



(a) **Compare $\triangle OAP$ and $\triangle OBP$:**

- $OA = OB$ (Both are **radii**).
- $\angle OAP = \angle OBP = 90°$ (Tangent meets radius at $90°$).
- OP is common to both (The **Hypotenuse**).

(b) **Condition:** We have a Right angle, a Hypotenuse, and a Side. **RHS Congruence.**

(c) **Conclusion:** Since the triangles are congruent, all corresponding sides must be equal. Therefore, side PA corresponds to side PB.

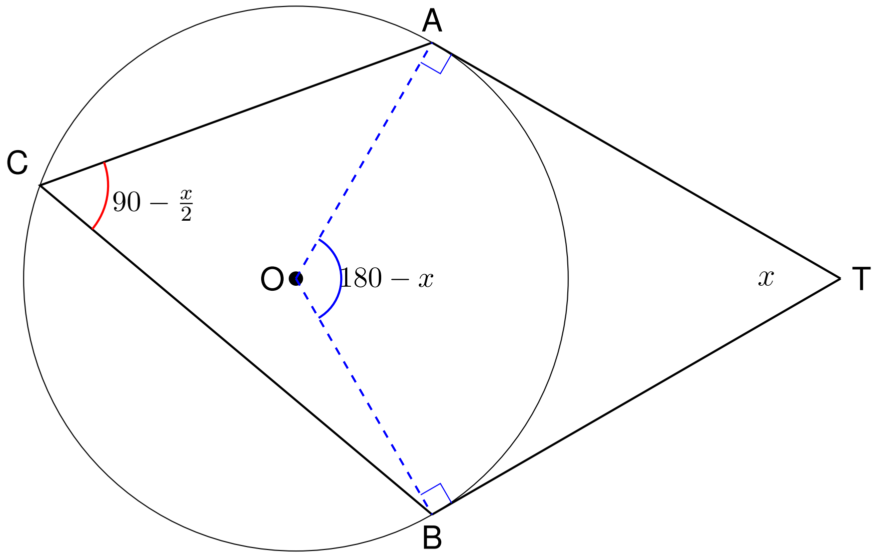
$$\boldsymbol{PA = PB}$$

Q.E.D.

4) Exam Application: Algebraic Proof

Pro-Tip: Use Algebra

When asked to "Show that...", it is usually best to assign a letter (like x) to the angle mentioned in the question, and work forward from there.



Let $\angle ATB = x$.

Step 1: Find Angle at Centre ($\angle AOB$) Consider the quadrilateral $OATB$.

- Angles $\angle OAT$ and $\angle OBT$ are both $90°$ (Tangent meets radius).
- Angles in a quadrilateral sum to $360°$.

$$\angle AOB = 360 - 90 - 90 - x = \mathbf{180 - x}$$

Step 2: Find Angle at Circumference ($\angle ACB$) The angle at the centre is twice the angle at the circumference.

$$\angle AOB = 2 \times \angle ACB$$
$$180 - x = 2 \times \angle ACB$$
$$\angle ACB = \frac{180 - x}{2}$$
$$\angle ACB = 90 - \frac{x}{2}$$

Conclusion: Since $x = \angle ATB$, we have proved:

$$\angle ACB = 90° - \frac{1}{2}\angle ATB$$

Q.E.D.